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# Counterexamples and uniqueness for $L^p(\partial\Omega)$ oblique derivative problems

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## Abstract

Harmonic functions defined in Lipschitz domains of the plane that have gradient nontangentially in  $L^2$  and have nonnegative oblique derivative almost everywhere on the boundary with respect to a continuous transverse vector field are shown to be constant. Explicit examples that have almost everywhere vanishing oblique derivative are constructed when  $L^2$  is replaced by  $L^p$ ,  $p < 2$ . Explicit examples with vanishing oblique derivative are constructed when  $p \leq 2$  and the continuous vector field is replaced by large perturbations of the normal vector field. Optimal bounds on the perturbation, depending on  $p \leq 2$  and the Lipschitz constant, are given which imply that only the constant solution has nonnegative oblique derivative almost everywhere. Examples are constructed in higher dimensions and the Fredholm properties of certain nonvariational layer potentials discussed.

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## 0. Introduction

Let  $\Omega \subset \mathbb{R}^n$  ( $n \geq 2$ ) be a bounded Lipschitz domain. Denote by  $N = N_Q$  the outer unit normal vector at points  $Q \in \partial\Omega$ . The normal vector exists with respect to surface measure  $ds$  at

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almost every point of the boundary. Let  $\vec{\alpha}$  denote a unit vector field that is continuous on  $\partial\Omega$  and transverse to  $\partial\Omega$ , meaning that there is a constant  $c > 0$  so that

$$\vec{\alpha}(Q) \cdot N_Q \geq c > 0 \quad a.e.(ds(Q)). \quad (0.1)$$

The homogeneous oblique derivative problem for solutions to Laplace's equation  $\Delta h = 0$  can be formulated as follows. Given  $\Omega$ ,  $\vec{\alpha}$  and  $1 \leq p \leq \infty$ , a harmonic function  $h$  defined in  $\Omega$  is said to be a solution to the  $L^p$  homogeneous oblique derivative problem if

$$(i) \quad \nabla h^* \in L^p(\partial\Omega)$$

and

$$(ii) \quad \text{n.t.} \lim_{X \rightarrow Q} \vec{\alpha}(Q) \cdot \nabla h(X) = 0 \quad a.e.(ds(Q)).$$

Here  $*$  denotes the nontangential maximal function (of  $\nabla h$ ) and n.t. lim denotes the nontangential limit. The Lebesgue spaces are with respect to surface measure. In this article we are interested in the dimension of the space of solutions to the homogeneous oblique derivative problem for various  $p$ .

In Giraud's classical formulation [12] the domain is assumed to be of class  $C^2$ ,  $h \in C(\overline{\Omega})$ , and if the (classical) derivative exists and equals zero at each point of the boundary, then  $h$  is identically constant. The proof is based on the Hopf maximum principle (see [2, p. 154], for example).

Most recently N.S. Nadirashvili [19,20] generalized this approach to the problem to Lipschitz domains and transverse vector fields not necessarily continuous. Again if the oblique derivative of  $h \in C(\overline{\Omega})$  vanishes at each point of the boundary, then  $h$  is constant.

The formulation here comes from an article of A.P. Calderón's [4, pp. 43, 44] which uses certain nonvariational layer potentials in order to solve a nonhomogeneous boundary value problem for the oblique derivative of harmonic functions. It will be stated after defining some terminology.

A nontangential approach region for each  $Q \in \partial\Omega$  is defined by

$$\Gamma(Q) = \Gamma_a(Q) = \{X \in \Omega: \text{dist}(X, \partial\Omega) > a|X - Q|\},$$

where  $0 < a < 1$  is fixed small enough depending on the Lipschitz nature of the domain. Given a function  $F$  in  $\Omega$  the nontangential limit at  $Q$ , if it exists, is defined by

$$\text{n.t.} \lim_{X \rightarrow Q} F(X) = \lim_{X \in \Gamma(Q), X \rightarrow Q} F(X).$$

For such limits to exist it is generally required to have some type of control on a corresponding nontangential maximal function defined by

$$F^*(Q) = \sup_{X \in \Gamma(Q)} |F(X)|.$$

The phrase " $F$  is nontangentially in  $L^p(\partial\Omega)$ " will mean that  $F^* \in L^p(\partial\Omega)$  and has nontangential limits  $a.e.(ds)$ .

**Theorem 0.1.** (See Calderón [4].) Let  $\Omega \subset \mathbb{R}^n$  ( $n \geq 2$ ) be a bounded Lipschitz domain with connected complement. Let  $\vec{\alpha}$  be a continuous transverse (0.1) unit vector field on  $\partial\Omega$ . Then there exist a finite number of linearly independent functions  $f_1, \dots, f_N \in L^2(\partial\Omega)$  so that if  $g \in L^2(\partial\Omega)$  satisfies

$$\int_{\partial\Omega} g f_j ds = 0 \quad (j = 1, \dots, N) \quad (0.2)$$

then there exists a harmonic function  $u$  in  $\Omega$  with  $\nabla u$  nontangentially in  $L^2(\partial\Omega)$  such that

$$\text{n.t.} \lim_{X \rightarrow Q} \vec{\alpha}(Q) \cdot \nabla u(X) = g(Q) \quad \text{a.e.}(ds(Q)).$$

Furthermore  $u$  will be unique up to the addition of  $N$  solutions to the homogeneous oblique derivative problem, i.e. the dimension of the space of solutions to the homogeneous problem for  $p = 2$  equals the number of linear conditions (0.2) imposed on  $g$  for solvability.

Calderón proves this result for an interval of  $p$ 's about 2,  $p(\Omega) < p < \frac{p(\Omega)}{p(\Omega)-1}$  where  $p(\Omega)$  depends only on the Lipschitz geometry of the domain. C.E. Kenig, J. Pipher [16] and J. Pipher [23], by methods necessarily different than Calderón's integral equation method (see Section 4), extended this formulation of the problem to  $p < \infty$ , BMO and Hölder spaces.

An issue raised by these theorems is, what are the finite number of linear conditions when data must necessarily be taken on at almost every point rather than at each point? Or at least, how many linear conditions are there? What are some examples of nonconstant solutions to the homogeneous problem?

For  $p < 2$  and  $n \geq 2$  Calderón's theorem, together with known examples of harmonic functions with vanishing Dirichlet data on reentrant corners, already suffices to imply that the  $L^p$  homogeneous oblique problem for a Lipschitz domain and continuous vector field can have an infinite number of linearly independent solutions. This is explained in Section 4. The Fredholm and non-Fredholm properties of Calderón's nonvariational potentials are also discussed there.

In Section 1 for any  $p < 2$  a domain (1.9) and a continuous vector field (1.13) with one non-constant solution (1.12) are given explicitly. This *elementary example* is generalized in that same section to examples with an arbitrary *finite* number of solutions to the homogeneous problem. The domains are  $C^1$  curvilinear polygons. The domains, vector fields and solutions may be made as explicit as one wishes by, for example, choosing circular arcs for the polygonal sides. A lemma on interpolation of finite Blaschke products, and what seems to be a novel definition of a certain kind of matrix are incidental features of this section.

The explicit examples of Section 1, more than those of the later section, clarify the relationships between solutions to the homogeneous problem and the transverse vector field. One benefit of this is that the construction, when attempted in the most general way for a Lipschitz boundary, leads to a contradiction if the gradient of a solution is hypothesized to be nontangentially in  $L^2$ . This is done in Section 2, proving Theorem 2.1 which says that for such harmonic functions in a planar bounded simply connected Lipschitz domain, and any continuous transverse vector field, there is only the constant solution to the homogeneous problem. The proof of Section 2 may be read without reference to the explicit examples of the previous section.

By Calderón's theorem then, there is *one* linear condition to be met by  $L^2$  oblique data when in the planar setting. This condition is identified in Section 3 as orthogonality to a unique *nonnegative* function. In order to prove this, Corollary 3.1, saying that any solution that takes *nonnegative*

data must be constant, is established by making more complete use of the argument used in Section 2.

By generalizing the construction of the elementary example from Section 1 in another way, nontrivial solutions to the  $L^2$  homogeneous problem with respect to certain *discontinuous* vector fields are written down in Section 5. The vector fields are perturbations of the normal vector field. Theorem 5.1 and Corollary 5.2 give a condition on the size of the perturbation, depending on  $1 < p \leq 2$  and the Lipschitz constant for  $\Omega$ , that implies a solution with gradient nontangentially in  $L^p$  and nonnegative oblique data must be constant.

The results here and in [33] are motivated by two considerations. In [32] classical layer potentials and a variational Neumann problem for the biharmonic equation in Lipschitz domains of  $\mathbb{R}^n$  are identified. The theorems proved there for a 4th order equation turn out to be precise analogues to earlier results for the Laplacian [13,30], opening up the possibility for an integral equation method for elliptic equations of all orders on Lipschitz boundaries. However, in the general higher order case one encounters a *coerciveness* problem on the boundary because the classical layer potentials are tied to the variational Neumann problem (see [24] for the coercive estimates on Lipschitz boundaries for the Dirichlet problem). Coercive estimates for the Neumann problem are necessarily stronger than those for the Dirichlet problem (Gårding's inequality in the classical setting). To get around this problem one is tempted to modify layer potentials based on the normal vector field by substituting a transverse field with more differentiability. In effect one attempts to replace coercive estimates for the Neumann problem with those for the Dirichlet problem. A second order model for this is Calderón's nonvariational potential method. The uniqueness results for that method, however, have not been quite as precise as those that can be obtained with classically defined potentials. This leads to the second consideration. Lipschitz domain methods, including layer potentials, can also be used to study boundary value problems in compact polyhedra of  $\mathbb{R}^n$  [31,34,35]. Interesting problems arise in that setting because boundaries (even those homeomorphic to the sphere) need not be given as graphs in rectangular coordinate systems (other topological anomalies also arise [5,7]). However, in compact polyhedra it would seem unacceptable to solve problems modulo a finite number of linear conditions unless the number were quantified in terms of the geometry of the polyhedron or shown to be independent of it.

Some problems are posed in the final section.

## 1. Explicit counterexamples for $p < 2$

### 1.1. The elementary example

Given  $p < 2$  a bounded domain  $\Omega$  in the complex plane with piecewise smooth boundary between two reentrant corners will be constructed along with a Hölder continuous vector field  $\vec{\alpha}$  and nontrivial solution  $h$  to the corresponding homogeneous oblique derivative problem.

The vector field  $(\alpha_1, \alpha_2)$  may be identified with the complex numbers  $\alpha = \alpha_1 + i\alpha_2$ . (Half) the gradient of a real harmonic  $h$  may be identified with  $\bar{\partial}h$ , the complex conjugate of an analytic function, where  $\bar{\partial} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$ . It will be arranged that

$$\alpha = \pm i \frac{\bar{\partial}h}{|\partial h|} \quad a.e.(ds) \quad (1.1)$$

so that (1)  $\vec{\alpha} \cdot \nabla h = 2 \operatorname{Re}(\alpha \partial h) = 0$  a.e.( $ds$ ), (2)  $\alpha$  is transverse, and (3)  $\alpha$  is continuous on  $\partial\Omega$ . As shown in Lemma 2.7: it is *necessary* to use both signs in (1.1). The change in sign occurs at the two corners.

The domain  $\Omega$  will be defined by defining a conformal map  $\Phi$  from the open unit disc  $\mathbb{D}$ . Then the composition  $H = h \circ \Phi$  will be harmonic in  $\mathbb{D}$  and

$$\partial H = \partial h \circ \Phi \Phi'. \quad (1.2)$$

The transverse condition for  $\alpha$  on  $\partial\Omega$  together with the sign change of (1.1), when brought back to  $\partial\mathbb{D}$ , implies that the *imaginary part* of the product of analytic functions

$$F(z) = z \partial H(z) \quad (1.3)$$

must change sign at the points of  $\partial\mathbb{D}$  that are mapped to the corner points of  $\Omega$ . See (2.8) and the proof of Lemma 2.7. By (1.1) and (1.2) the direction of  $\frac{\partial H}{\Phi}$  must reverse itself *diametrically* at the preimage corner points in order to insure the continuity of (1.1). In other words,  $\partial H$  and thus  $F$  must change direction on  $\partial\mathbb{D}$  in a *nondiametric* way so that the diametric reversal of direction is effected by mapping to a corner of  $\Omega$ . Moreover, a diametric reversal on  $\partial\mathbb{D}$  interferes with desired  $p < 2$  integrability properties, as will be seen.

Given these requirements on simultaneous change of sign of the imaginary part of  $F$  and change in direction of  $F$  as  $F$  takes values on  $\partial\mathbb{D}$ , one is led to consider  $F$  as a mapping of  $\partial\mathbb{D}$  that will pass through the real axis only at 0 or  $\infty$ . An  $F$  with imaginary sign changes at  $\infty$  will be used.

For  $0 < b < 1$  define

$$F_b(z) = \sinh\left(b \log \frac{1+z}{1-z}\right), \quad z \in \bar{\mathbb{D}}, \quad z \neq \pm 1. \quad (1.4)$$

The disc is thus mapped to the strip between  $\pm ib\frac{\pi}{2}$  which is then given angles at  $\infty$  by composition with  $\sinh$ . Furthermore  $F_b(0) = 0$  as in (1.3). By (1.1)–(1.3) one is led to consider the vector field

$$A = \frac{iz\bar{F}_b}{|F_b|}$$

discontinuous on  $\partial\mathbb{D}$  at  $\pm 1$ , but transverse to  $\partial\mathbb{D}$  up to sign change. In fact

$$\operatorname{Re}\left(\frac{iz\bar{F}_b}{|F_b|}\bar{z}\right) = \operatorname{Im} \frac{F_b}{|F_b|} \begin{cases} > \sin b\frac{\pi}{2}, & \operatorname{Im} z > 0, \\ < -\sin b\frac{\pi}{2}, & \operatorname{Im} z < 0, \end{cases} \quad (1.5)$$

since the image of  $\partial\mathbb{D} \setminus \{\pm 1\}$  under  $F_b$  is the hyperbola

$$\frac{y^2}{\sin^2 b\frac{\pi}{2}} - \frac{x^2}{\cos^2 b\frac{\pi}{2}} = 1. \quad (1.6)$$

The vector field  $A$  restricted to the upper half circle has limits  $e^{i(1-b)\frac{\pi}{2}}$  at 1 and  $e^{i(1+b)\frac{\pi}{2}}$  at  $-1$ . The vector field  $-A$  restricted to the lower half circle has limits  $e^{i(b-1)\frac{\pi}{2}}$  at 1 and  $e^{-i(1+b)\frac{\pi}{2}}$  at  $-1$ . Thus the vector field

$$\alpha \circ \Phi = \begin{cases} \frac{A\Phi'}{|\Phi'|}, & \operatorname{Im} z > 0, \\ \frac{-A\Phi'}{|\Phi'|}, & \operatorname{Im} z < 0 \end{cases} \quad (1.7)$$

can be extended continuously to the points  $\pm 1$  if a conformal map  $\Phi$  can be defined to be piecewise  $C^1$  on  $\partial\mathbb{D}$  with

$$\lim_{\theta \downarrow 0} \frac{\Phi'(e^{i\theta})}{|\Phi'(e^{i\theta})|} = ce^{i(b-1)\frac{\pi}{2}}, \quad \lim_{\theta \uparrow 0} \frac{\Phi'(e^{i\theta})}{|\Phi'(e^{i\theta})|} = ce^{i(1-b)\frac{\pi}{2}} \quad (1.8)$$

for a constant  $|c| = 1$ , and similarly at the point  $-1$ . The map can be defined by composing elementary functions. One example is

$$\Phi(z) = \Phi_b(z) = \left[ \left( \frac{1-z}{1+z} \right)^{2-b} + 1 \right]^{-1} = 1 - \left[ \left( \frac{1+z}{1-z} \right)^{2-b} + 1 \right]^{-1}. \quad (1.9)$$

The first formula conveniently yields (1.8) with  $c = 1$  and the second can similarly be used to verify conditions at the point  $-1$ . Thus (1.7) is continuous on  $\partial\mathbb{D}$ .

Define  $\Omega = \Phi(\mathbb{D})$ . It is the union of two discs with centers at  $(2\cos(1-b)\frac{\pi}{2})^{-1}e^{\pm i(1-b)\frac{\pi}{2}}$  and with the corresponding circles each passing through 0 and 1. From (1.7)–(1.9)  $\alpha$  is continuous on  $\partial\Omega$ . The transverseness of  $\alpha$  on  $\partial\Omega$  reduces to (1.5) by forming the inner product with the normal vectors  $\frac{e^{i\theta}\Phi'(e^{i\theta})}{|\Phi'(e^{i\theta})|}$  at  $\Phi(e^{i\theta})$ .

The gradient of the as yet undefined  $h$  will be nontangentially in  $L^p(\partial\Omega)$  if and only if the analytic function

$$z^{-1}F_b(z)(\Phi'(z))^{1/p-1} \quad (1.10)$$

is in the classical Hardy space  $H^p(\mathbb{D})$ . See Sections 2.2 and 2.7. Writing  $2F_b(z) = (\frac{1+z}{1-z})^b - (\frac{1-z}{1+z})^b$  verification near the point 1 reduces to checking that

$$(1-z)^{-b-(1-b)(p-1)/p} \in H^p(\mathbb{D}). \quad (1.11)$$

This occurs when  $p < 2 - b$ . Substituting  $z = \Phi^{-1}(w)$  into  $\frac{F_b}{z\Phi'}$  yields

$$G'_b(w) = \frac{1}{(2-b)(w(1-w))^{\frac{1}{2-b}}} \frac{w^{\frac{2b}{2-b}} - (1-w)^{\frac{2b}{2-b}}}{w^{\frac{2}{2-b}} - (1-w)^{\frac{2}{2-b}}} \quad (1.12)$$

for an analytic function  $G_b$  in the simply connected domain  $\Omega$ . Define

$$h = \operatorname{Re} G_b.$$

Then  $\nabla h$ , which can be identified with  $\overline{G'_b}$ , is nontangentially in  $L^p(\partial\Omega)$  for  $p < 2 - b$ . The vector field

$$\alpha(w) = \begin{cases} i \frac{\overline{G'_b}}{|G'_b|}, & w \in \partial\Omega, \operatorname{Im} w > 0, \\ -i \frac{\overline{G'_b}}{|G'_b|}, & w \in \partial\Omega, \operatorname{Im} w < 0 \end{cases} \quad (1.13)$$

extends to be continuous and transverse. It extends, moreover, to be Lipschitz continuous.

In sum:

*Given  $p < 2$  choose  $0 < b < 2 - p$ . Let  $\Omega$  be the Lipschitz domain (curvilinear polygon) conformally equivalent to the unit disc under the map (1.9). There is a real harmonic function  $h$  with gradient equal to  $\overline{G'_b}$  (1.12) and, on  $\partial\Omega$ , a Lipschitz continuous transverse unit vector field  $\alpha$  (1.13) so that  $h$  is a solution to the  $L^p(\partial\Omega)$  homogeneous oblique derivative problem with respect to  $\alpha$ .*

## 1.2. Solution spaces of arbitrary finite dimension

In order to construct a homogeneous oblique derivative problem with some finite number of independent solutions a domain with more corners is used. Taking a slightly different tact, a conformal equivalence  $\Phi: \mathbb{D} \rightarrow \Omega$  will be assumed given, where  $\Omega$  is a bounded piecewise  $C^1$  domain with  $2N$  reentrant corners all with interior angle  $\pi(2 - b)$ . The boundary will be smooth ( $C^1$ ) at every other point. More precisely there is a neighborhood in  $\partial\Omega$  about each corner point that is the image of a piecewise  $C^1$  curve  $z: (-1, 1) \rightarrow \partial\Omega$  so that  $z(0)$  is the corner point,  $z'(t)$  is nonvanishing, continuous on  $(-1, 0]$  and continuous on  $[0, 1)$  with

$$\arg z'(0-) - \arg z'(0+) = \pi(1 - b). \quad (1.14)$$

The conformal map extends  $\Phi: \overline{\mathbb{D}} \rightarrow \overline{\Omega}$  to be a homeomorphism by a theorem of Carathéodory. Define numbers

$$0 = s_1 < t_1 < s_2 < t_2 < \cdots < s_N < t_N < 2\pi = s_{N+1}$$

and

$$\mu_j = e^{is_j}, \quad v_j = e^{it_j} \quad (j = 1, \dots, N). \quad (1.15)$$

It will be assumed that  $\Phi$  maps  $\{\mu_1, v_1, \dots, v_N\}$  onto the  $2N$  corners of  $\Omega$ .

It will be shown below that there is a finite Blaschke product  $B$  in  $\mathbb{D}$  such that  $B(z) = 1$  if and only if  $z \in \{\mu_1, \dots, \mu_N\}$ , and  $B(z) = -1$  if and only if  $z \in \{v_1, \dots, v_N\}$ . If  $z_0$  is a zero of  $B$  let  $\phi(z)$  be the Möbius transformation that maps 0 to  $z_0$  and 1 to 1. Then  $B \circ \phi$  is a finite Blaschke product that maps 0 to 0 and takes the values  $\pm 1$ , as before, precisely at the points that are mapped by  $\Phi \circ \phi$  to the corners of  $\Omega$ . Consequently it may be assumed without loss of generality that  $\Phi$  was such that the Blaschke product  $B$  to be obtained from Lemma 1.3 has the additional property  $B(0) = 0$ .

Assuming such a  $B$  then, define

$$F(z) = \sinh\left(b \log \frac{1+B(z)}{1-B(z)}\right).$$

By the above mentioned properties of  $B$ ,  $F$  maps  $\partial\mathbb{D}$  in the counterclockwise direction, starting at 1, onto the hyperbola (1.6), starting at the upper right, a total of  $N$  times. As in (1.5)

$$\operatorname{Im} \frac{F}{|F|}(e^{i\theta}) \begin{cases} > \sin b \frac{\pi}{2}, & s_j < \theta < t_j, \\ < -\sin b \frac{\pi}{2}, & t_j < \theta < s_{j+1}, \end{cases} \quad (1.16)$$

$j = 1, \dots, N$ . Letting  $v-$  denote the limit at any one of the  $v_j$  taken in the counterclockwise direction on  $\partial\mathbb{D}$ , and  $v+$  the limit in the opposite direction, it likewise follows that

$$\arg F(v-) - \arg F(v+) = \pi - \frac{\pi}{2}b - \left(\pi + \frac{\pi}{2}b\right) = -\pi b \quad (1.17)$$

and similarly

$$\arg F(\mu-) - \arg F(\mu+) = -\pi b$$

by again inspecting the hyperbola.

A corner point of  $\partial\Omega$ , say  $\Phi(v)$ , may always be assumed to be at the origin with the two tangent vectors from (1.14)  $z'(0-) = e^{-i\frac{\pi}{2}b}$  and  $z'(0+) = e^{-i\frac{\pi}{2}(2-b)}$ . Then in a neighborhood of  $v$  in  $\mathbb{D}$  the composition  $\Phi^{\frac{1}{2-b}}(z)$  is a conformal equivalence that extends to a homeomorphism of an open arc of  $\partial\mathbb{D}$  containing  $v$  to a  $C^1$  arc containing  $\Phi(v)$ . By a theorem of Lindelöf ([17, p. 48], [25, p. 44])  $\arg(\Phi^{\frac{1}{2-b}})'$  extends to the arc continuously. In this way it can be concluded that

$$\arg \Phi'(v-) - \arg \Phi'(v+) = \pi(1-b) \quad (1.18)$$

just as for  $z'(0\pm)$ . The same also holds at any of the  $\mu_j$ . By (1.17) and (1.18) it follows that

$$\arg(\bar{F}\Phi')(v-) - \arg(\bar{F}\Phi')(v+) = \pi b + \pi(1-b) = \pi$$

and the same at any  $\mu_j$ . Consequently as in (1.7) and (1.13)

$$\alpha \circ \Phi(e^{i\theta}) = \begin{cases} i \frac{|\Phi'|}{|F|} \left( \frac{F}{e^{i\theta}\Phi'} \right), & s_j < \theta < t_j, \\ -i \frac{|\Phi'|}{|F|} \left( \frac{F}{e^{i\theta}\Phi'} \right), & t_j < \theta < s_{j+1} \end{cases} \quad (1.19)$$

for  $j = 1, \dots, N$  is a continuous vector field on  $\partial\mathbb{D}$ . The transverseness of  $\alpha$  on  $\partial\Omega$  follows from (1.16) as in the elementary example.

Because  $F(0) = 0$  and because  $\Omega$  is simply connected there is a real harmonic function  $h$  with

$$\partial h = \left( \frac{F}{z\Phi'} \right) \circ \Phi^{-1} \quad (1.20)$$



in  $\Omega$ . Given  $0 < b < 1$  choose  $p$  so that  $1 < p < \frac{2}{1+b}$ . Then  $q = \frac{1}{p-1} > \frac{1+b}{1-b}$  has dual exponent  $q' = \frac{1}{2-p}$ . Showing that  $\nabla h$  is nontangentially in  $L^p(\partial\Omega)$  is again equivalent to showing that (1.10) is in  $H^p(\mathbb{D})$ . By following arguments due to A.P. Calderón (see (2.4) and Section 2.4)  $\Phi'^{-1} \in H^1(\mathbb{D})$ . Applying Hölder's inequality

$$\int_{\partial\mathbb{D}} |F|^p |\Phi'|^{1-p} d\theta \leq \|\Phi'^{-1}\|_1^{p-1} \left( \int_{\partial\mathbb{D}} |F|^{\frac{p}{2-p}} d\theta \right)^{2-p}.$$

By the construction below of the Blaschke product defining  $F$ ,  $|F|$  behaves at each of the  $\mu_j$  and  $\nu_j$  as does  $|1 - z|^{-b}$  at the point 1. Consequently  $\partial h$  is nontangentially in  $L^p(\partial\Omega)$  if  $bp < 2 - p$  which is how  $p$  was chosen. The exponent of integrability  $p$  can be chosen arbitrarily close to 2 by choosing  $b$ .

With the existence of  $B$  yet to be established

**Theorem 1.1.** *Given  $0 < b < 1$ , let  $\Omega \subset \mathbb{R}^2$  be a bounded simply connected piecewise  $C^1$  domain with  $2N$  corners each of opening  $\pi(2 - b)$ , and with the boundary  $C^1$  in a neighborhood of every other point. Then there is a transverse and continuous unit vector field  $\vec{\alpha}$  defined on  $\partial\Omega$  so that the associated homogeneous oblique derivative problem has  $N$  linearly independent solutions each with gradient nontangentially in  $L^p(\partial\Omega)$  for any  $p < \frac{2}{1+b}$ .*

**Proof.** The vector field is defined in (1.19). The analytic function (1.20) provides one solution  $h$  with the correct integrability properties and so that on  $\partial\Omega$

$$\frac{1}{2} \vec{\alpha} \cdot \nabla h = \operatorname{Re}(\alpha \partial h) = \operatorname{Re} \left( \alpha \left( \frac{F}{z\Phi'} \right) \circ \Phi^{-1} \right) = 0 \quad a.e.(ds)$$

or equivalently

$$\operatorname{Re} \left( \alpha \circ \Phi \frac{F}{e^{i\theta} \Phi'} \right) = 0 \quad a.e.(d\theta). \quad (1.21)$$

Other solutions must likewise arise from analytic functions in  $\mathbb{D}$  and have the same arguments  $a.e.(d\theta)$  as does the analytic function  $G_1(z) = \frac{F(z)}{z\Phi'(z)}$  in (1.21).

By its construction below,  $B$  has  $N$  zeros in  $\mathbb{D}$  counting by multiplicity, as then does  $F$ , so that  $G_1$  has  $N - 1$  zeros in  $\mathbb{D}$ . Suppose  $G_1(z_1) = 0$ . Let  $\phi_1(z) = \frac{z - z_1}{1 - \bar{z}_1 z}$ . Define  $g_2(z) = \frac{G_1(z)}{\phi_1(z)}$ . Then  $g_2$  is analytic in  $\mathbb{D}$  with one fewer zero than  $G_1$ . It has the same integrability properties as  $G_1$ , vis.  $g_2 \in L^p(\partial\mathbb{D}, |\Phi'| d\theta)$ .

On  $\partial\mathbb{D}$

$$\arg((\phi_1 + \bar{\phi}_1 + 2)G_1) = \arg G_1 \quad a.e.(d\theta)$$

and  $(\phi_1 + \bar{\phi}_1 + 2)G_1$  extends analytically inside  $\mathbb{D}$  as

$$(\phi_1 + 2)G_1 + g_2 = G_2$$

since  $\phi_1 \bar{\phi}_1 = 1$  on  $\partial \mathbb{D}$ . The analytic  $G_2$  is in  $L^p(\partial \mathbb{D}, |\Phi'| d\theta)$ , has  $N - 2$  zeros, and satisfies the required  $\arg G_2 = \arg G_1$  a.e. ( $d\theta$ ). Consequently it yields an independent solution  $h_2$  to the homogeneous oblique derivative problem with respect to  $\vec{\alpha}$ . Continuing, finishes the proof.  $\square$

**Remark 1.2.** There are more solutions. For example  $(-i\phi_1 + 2)G_1 + ig_2$  yields a solution.

### 1.3. An interpolation result

Cantor and Phelps [6] showed how to define a *finite* Blaschke product that takes on a finite number of prescribed values of modulus one at a finite number of prescribed points on the unit circle. However, because of the generality in which they work their method cannot provide a property needed here. It is necessary for the continuity of the oblique vector field that the finite Blaschke product map *no other points* of the circle to  $\pm 1$ . For example, a (finite) Blaschke product that maps 1 and  $i$  to 1, and maps  $-1$  and  $-i$  to  $-1$  must necessarily map other points to  $\pm 1$ . In contrast, the problem solved here concerns alternating the values  $\pm 1$  as the circle is traversed. Belna, Carroll and Piranian [1, Theorem 1, p. 697] showed that when there is but one prescribed value a Blaschke product exists that takes on that value at a prescribed countable  $G_\delta$  set and nowhere else. The idea of initially mapping the disc to the half-plane and the  $v_j$  to  $\infty$  is borrowed from them.

**Lemma 1.3.** *Given the numbers (1.15) on the unit circle, there is a finite Blaschke product  $B(z)$  with exactly  $N$  zeros in  $\mathbb{D}$ , counted by multiplicity, so that for  $z \in \bar{\mathbb{D}}$ ,  $B(z) = 1$  if and only if  $z \in \{\mu_1, \dots, \mu_N\}$ , and  $B(z) = -1$  if and only if  $z \in \{v_1, \dots, v_N\}$ .*

The remainder of this subsection is devoted to the proof.

1.3.1. Let  $\mathcal{C}_N$  denote the collection of  $N \times N$  matrices  $A = [\vec{a}_1 \cdots \vec{a}_j \cdots \vec{a}_N] = [a_{lj}]$  satisfying

$$a_{jj} > a_{j+1,j} > \cdots > a_{Nj} > a_{1j} > \cdots > a_{j-1,j} \quad (j = 1, \dots, N) \quad (1.22)$$

(circular monotonicity of the columns) and

$$\sum_{l=1}^N a_{lj} = 0 \quad (j = 1, \dots, N) \quad (1.23)$$

(column sum zero).

1.3.2.

**Lemma 1.4.** *Given any nonzero real vector  $\vec{v}$  satisfying  $v_1 + \cdots + v_N = 0$ , there exists an index  $j$  depending only on  $\vec{v}$  so that for every  $A \in \mathcal{C}_N$*

$$\vec{v} \cdot \vec{a}_j > 0.$$

**Proof.** Define partial sums  $S_k = v_1 + \cdots + v_k$  ( $k = 1, \dots, N$ ). Let  $m$  be the smallest index so that  $S_m = \min_{1 \leq k \leq N} \{S_k\}$ . By hypothesis  $S_m \leq 0$ .

For convenience define  $v_{N+l} = v_l$  and  $a_{N+l,j} = a_{l,j}$  ( $l = 1, \dots, N$ ). Define the sums  $V_k = v_{m+1} + \cdots + v_{m+k}$  ( $k = 1, \dots, N-1$ ). Then  $V_k = S_{k+m} - S_m \geq 0$  for  $k \leq N-m$ ,  $V_k = S_{k+m-N} - S_m > 0$  for  $N-m+1 \leq k$ , and at least one  $V_k$  is positive.

Summation by parts yields

$$\vec{v} \cdot \vec{a}_j = \sum_{k=1}^N v_{m+k} a_{m+k,j} = \sum_{k=1}^{N-1} V_k (a_{m+k,j} - a_{m+1+k,j}).$$

If  $j = m+1$  ( $j = 1$  if  $m = N$ ), then by circular monotonicity (1.22) the sum is positive.  $\square$

1.3.3. A vector  $\vec{x}$  is called *positive* if each component is positive  $x_j > 0$  ( $j = 1, \dots, N$ ). Denote by  $\vec{1}$  the  $N$ -vector  $(1, 1, \dots, 1)$ .

**Theorem 1.5.** If  $A \in \mathcal{C}_N$  then  $A$  has a positive eigenvector corresponding to the eigenvalue 0.

**Proof.** Let  $G$  denote the set of all proper convex combinations of the columns of  $A$ , i.e. all  $\vec{w} = t_1 \vec{a}_1 + \cdots + t_N \vec{a}_N$  where  $t_1 + \cdots + t_N = 1$  and  $t_j > 0$  for all  $j$ .  $G$  is convex and contained in the hyperplane of  $\mathbb{R}^N$  through the origin that is perpendicular to  $\vec{1}$ . If in addition  $G$  were contained in a flat or linear variety of dimension less than  $N-1$  then the lemma would be contradicted. Thus the column vectors are in general position and  $G$  is an open set in the hyperplane. Suppose the origin is not contained in  $G$ . Then by the convexity of  $G$  and  $\{\vec{0}\}$  (see, for example, [29, Theorem 2.9]) there is a vector  $\vec{v}$  in the hyperplane such that the closed half-space  $\{\vec{w}: \vec{v} \cdot \vec{w} \leq 0\}$  contains  $G$  and thus every column vector, contradicting the lemma.  $\square$

**Corollary 1.6.** Let  $A$  be an  $N \times N$  matrix that satisfies the circular monotonicity condition (1.22). Then there is a positive vector  $\vec{x}$  and a real number  $c$  such that  $A\vec{x} = c\vec{1}$ .

**Proof.** The matrix

$$\tilde{A} = A - \frac{1}{N} [(\vec{a}_1 \cdot \vec{1})\vec{1} \cdots (\vec{a}_N \cdot \vec{1})\vec{1}]$$

satisfies both conditions (1.22) and (1.23). Let  $\vec{x}$  be the positive eigenvector for  $\tilde{A}$ .  $\square$

1.3.4. End of the proof of Lemma 1.3

For a real number  $c$  and positive numbers  $x_1, \dots, x_N$  to be determined define

$$f(z) = x_1 \frac{v_1 + z}{v_1 - z} + \cdots + x_N \frac{v_N + z}{v_N - z} + ic.$$

Each  $x_j \frac{v_j + z}{v_j - z}$  maps  $\mathbb{D}$  onto the right half-plane, and for  $z = e^{i\theta}$  ( $t_j < \theta < t_j + 2\pi$ ) strictly decreases down the imaginary axis. Thus  $f(e^{i\theta})$  downwardly traverses the imaginary axis  $N$  times, once on each arc between successive  $v_1, \dots, v_N, v_1$ . By deleting small enough open discs from  $\mathbb{D}$  about each  $v_j$  it follows that  $f$  maps the resulting boundary a total of  $N$  times outside of

$\{z: \operatorname{Re} z > 0 \text{ and } |z| < R\}$  for any given  $R$ . Consequently  $f$  maps  $\mathbb{D}$  onto the right half-plane. By the argument principle  $f(z) - 1$  has exactly  $N$  zeros in  $\mathbb{D}$  counted by multiplicity.

The fractional linear transformation  $\frac{1-w}{1+w}$  maps the closed right half-plane and  $\infty$  onto  $\overline{\mathbb{D}}$ . Thus

$$B(z) = \frac{1 - f(z)}{1 + f(z)}$$

maps  $\overline{\mathbb{D}}$  onto  $\overline{\mathbb{D}}$  continuously with  $N$  zeros in  $\mathbb{D}$  and  $B(v_j) = -1$  for  $j = 1, \dots, N$ . Further  $B(e^{i\theta})$  traverses the entire circle exactly once between successive  $v_j$ .

At  $\mu_l = e^{is_l}$ ,

$$if(\mu_l) = -c + \sum_{j=1}^N \frac{\sin(t_j - s_l)}{1 - \cos(t_j - s_l)} x_j.$$

Let  $a_{lj} = \frac{\sin(t_j - s_l)}{1 - \cos(t_j - s_l)}$  ( $l, j = 1, \dots, N$ ). Then the  $a_{lj}$  satisfy the circular monotonicity (1.22). Therefore by the corollary there exists a real number  $c$  and  $x_1, \dots, x_N > 0$  so that  $f(\mu_l) = 0$  and therefore  $B(\mu_l) = 1$  for all  $l = 1, \dots, N$ .  $B$  takes the values  $\pm 1$  at no other points by its monotonicity properties.  $B$  is a finite Blaschke product [27, p. 209].

## 2. Uniqueness for the $L^2(\partial\Omega)$ problem

When it is required that the gradients of harmonic functions be nontangentially in the class  $L^2(\partial\Omega)$ , examples of nonuniqueness cannot be constructed.

**Theorem 2.1.** *Let  $\Omega \subset \mathbb{R}^2$  be a bounded simply connected Lipschitz domain and  $\vec{\alpha}$  a continuous transverse unit vector field defined on the boundary. If  $h$  is a harmonic function in  $\Omega$  with  $\nabla h^* \in L^2(\partial\Omega)$  and with oblique derivative vanishing nontangentially a.e.( $ds$ ), then  $h$  is constant.*

The proof by contradiction is the content of this section.

2.1. Suppose  $h$  is as described but not constant. Let  $\Omega$  be conformally equivalent to the unit disc by the mapping  $\Phi: \mathbb{D} \rightarrow \Omega$ .

2.2. For  $p > 0$  recall the classical Hardy spaces  $H^p(\mathbb{D})$  of analytic functions  $F(z)$  satisfying  $\sup_{r < 1} \int_0^{2\pi} |F(re^{i\theta})|^p d\theta < \infty$ . If  $F \in H^p(\mathbb{D})$  then  $F$  has nontangential limits, denoted  $F(\theta)$ , a.e.( $d\theta$ ) on  $\partial\mathbb{D}$ . Further  $F(\theta) \in L^p(\partial\mathbb{D})$  and  $\log|F(\theta)| \in L^1(\partial\mathbb{D})$  unless  $F(z)$  is identically zero [11, p. 17]. By Carathéodory's classical theorem  $\Phi$  extends continuously to  $\partial\mathbb{D}$ , and to a homeomorphism of  $\overline{\mathbb{D}}$  onto  $\overline{\Omega}$ . Since Lipschitz boundaries are rectifiable  $\Phi' \in H^1(\mathbb{D})$  and  $\frac{d}{d\theta}\Phi(\theta) (= \frac{d}{d\theta}\Phi(e^{i\theta})) = ie^{i\theta} \lim_{r \rightarrow 1} \Phi'(re^{i\theta}) = ie^{i\theta}\Phi'(\theta)$  a.e.( $d\theta$ ) [11, pp. 42–44]. Thus a subset of  $\partial\mathbb{D}$  has positive measure if its image in  $\partial\Omega$  does, and because  $\log|\Phi'(\theta)|$  is integrable, every image in  $\partial\Omega$  of a subset of  $\partial\mathbb{D}$  with positive measure has positive measure [11, p. 45].

2.3.  $\operatorname{Arg} z$  denotes the *principle value* of the *argument* of  $z$  with range  $-\pi < \operatorname{Arg} z \leq \pi$ .

**Lemma 2.2.** Let  $\Omega \subset \mathbb{R}^2$  be a bounded simply connected Lipschitz domain conformally equivalent to the unit disc by  $\Phi: \mathbb{D} \rightarrow \Omega$ . Then  $\arg \Phi'(z)$  can be defined as a single valued bounded harmonic function in  $\mathbb{D}$  with  $\arg \Phi'(0) = \text{Arg}(\Phi'(0))$  and nontangential limits  $\arg \Phi'(\theta) \in L^\infty(\partial\mathbb{D})$ . Also  $\log \Phi'(z) = \log|\Phi'(z)| + i \arg \Phi'(z) \in H^1(\mathbb{D})$  is single valued.

Furthermore there exist constants  $l > 0$  and  $\delta > 0$  depending only on the Lipschitz geometry of  $\Omega$  so that for every interval  $I \subset \partial\mathbb{D}$  of length  $l$

$$\text{ess sup}_I \arg \Phi' - \text{ess inf}_I \arg \Phi' < \pi - \delta. \quad (2.1)$$

**Proof.** For  $t > 0$  define analytic functions in  $\mathbb{D}$ , continuous in  $\overline{\mathbb{D}}$ ,

$$\Phi^t(z) = \begin{cases} \frac{\Phi(ze^{it}) - \Phi(z)}{itz}, & 0 \neq z \in \overline{\mathbb{D}}, \\ \frac{e^{it} - 1}{it} \Phi'(0), & z = 0. \end{cases} \quad (2.2)$$

As  $t \rightarrow 0$  these converge uniformly for  $|z| \leq r < 1$  to  $\Phi'(z)$ , and on  $\partial\mathbb{D}$  to  $\Phi'(\theta)$  a.e. Because the  $\Phi^t(z)$  has no zeros it follows by the argument principle [22, p. 130] that a single valued and harmonic  $\arg \Phi^t$  is well defined when normalized by

$$\arg \Phi^t(0) = \arctan\left(\frac{\sin t}{1 + \cos t}\right) + \text{Arg}(\Phi'(0))$$

and is continuous on  $\overline{\mathbb{D}}$ . That the  $\arg \Phi^t$  are uniformly bounded in  $t$  follows from the maximum principle and from the principle fact about bounded Lipschitz domains which is that the boundary is a *finite* union of graphs of Lipschitz functions defined in rectangular coordinates over compact intervals. By inspecting the changes in argument for each  $\Phi^t(z)$  when  $\Phi(z)$  is restricted to a graph this assertion is established. Statement (2.1) also follows uniformly for each  $\Phi^t$  in place of  $\Phi'$  in the same way. Letting  $t$  converge to zero establishes the lemma for  $\Phi'$ .  $\square$

2.4. When  $\Omega$  is a domain above a Lipschitz graph A.P. Calderón [3] argued that  $|\Phi'(\theta)|$  is in fact an  $A_2$  weight. Since this is a local condition the same should be true in the bounded case. Here is a proof.

**Lemma 2.3.** With  $\delta > 0$  as in Lemma 2.2 there is an analytic function  $\Psi$  in  $\mathbb{D}$  continuous and nonvanishing in  $\overline{\mathbb{D}}$  so that

$$-\frac{\pi}{2} + \frac{\delta}{2} < \arg \frac{\Phi'}{\Psi}(\theta) < \frac{\pi}{2} - \frac{\delta}{2}, \quad 0 \leq \theta < 2\pi.$$

**Proof.** For each  $\theta \in \mathbb{R}$  let  $I(\theta) \subset \partial\mathbb{D}$  denote an interval centered at  $e^{i\theta}$  and of length  $\frac{1}{2}l$  with  $l$  from Lemma 2.2. Define means

$$m(\theta) = \frac{1}{2} \left( \text{ess sup}_{I(\theta)} \arg \Phi' + \text{ess inf}_{I(\theta)} \arg \Phi' \right).$$

By Lemma 2.2,  $m(\theta)$  is a  $2\pi$ -periodic bounded function. Let  $0 \leq \psi \in C^\infty(\mathbb{R})$  have support in  $[-\frac{1}{4}, \frac{1}{4}]$  with integral one. The convolution  $\psi * m$  is smooth,  $2\pi$ -periodic and by (2.1) has the property that

$$-\frac{\pi}{2} + \frac{\delta}{2} < \arg \Phi'(\theta) - \psi * m(\theta) < \frac{\pi}{2} - \frac{\delta}{2}, \quad a.e.(d\theta).$$

The Hilbert transform  $\widetilde{\psi * m}$  defined on  $\partial\mathbb{D}$  is also smooth. Defining  $\Psi(z)$  to be the Poisson extension of  $e^{-\widetilde{\psi * m} + i\psi * m}$  in  $\mathbb{D}$ , the lemma follows.  $\square$

Consequently  $\frac{\Phi'}{\Psi} \in H^1(\mathbb{D})$  has positive real part. By the same argument as [3, p. 1325]

$$0 < \operatorname{Re} \frac{\Phi'}{\Psi} \leq \left| \frac{\Phi'}{\Psi} \right| \leq \csc\left(\frac{\delta}{2}\right) \operatorname{Re} \frac{\Phi'}{\Psi} \quad (2.3)$$

in  $\mathbb{D}$ . Because  $\arg \frac{\Psi}{\Phi'}$  is also bounded as in Lemma 2.3, (2.3) holds for  $\frac{\Psi}{\Phi'}$  in place of  $\frac{\Phi'}{\Psi}$ . Thus  $\frac{\Psi}{\Phi'} \in H^1(\mathbb{D})$  since  $|\frac{\Psi}{\Phi'}|$  is seen to be dominated in  $\mathbb{D}$  by a positive harmonic function. Consequently  $\operatorname{Re} \frac{\Psi}{\Phi'}$  too is the Poisson extension of its boundary values and the rest of Calderón's argument then puts  $|\frac{\Phi'}{\Psi}|$  in the Muckenhoupt class  $A_2(\partial\mathbb{D})$  with constant depending only on  $\delta$ . (Also for Calderón's proof above a Lipschitz graph see [14, p. 97] or, for a proof using the Helson–Szegő theorem [15, pp. 134, 135].) Now by the continuity and nonvanishing of  $\Psi$  it follows that

$$\Phi' \text{ and } \Phi'^{-1} \in H^1(\mathbb{D}) \quad \text{and} \quad |\Phi'| \in A^2(\partial\mathbb{D}). \quad (2.4)$$

2.5. Another consequence of Lemmas 2.2 and 2.3 is

**Theorem 2.4.** *Let  $\Omega \subset \mathbb{R}^2$  be a bounded simply connected Lipschitz domain conformally equivalent to the unit disc by  $\Phi: \mathbb{D} \rightarrow \Omega$ . Then there are constants  $l_1 > 0$  and  $R < 1$  depending only on the Lipschitz geometry of  $\Omega$  so that for every interval  $I \subset \partial\mathbb{D}$  of length  $l_1$  the sets  $\Lambda_r = \{\Phi(re^{i\theta}): e^{i\theta} \in I\}$ ,  $R < r < 1$ , are the graphs of (smooth) Lipschitz functions with respect to the same rectangular coordinate system and with uniform Lipschitz constants. They converge uniformly to  $\Phi(I)$ . The normalized vectors  $\frac{\Phi'(re^{i\theta})}{|\Phi'(re^{i\theta})|}$  converge pointwise a.e.( $d\theta$ ), and*

$$\lim_{r \rightarrow 1} \int_I \left| \frac{\Phi'(re^{i\theta})}{|\Phi'(re^{i\theta})|} - \frac{\Phi'(\theta)}{|\Phi'(\theta)|} \right|^p |\Phi'(\theta)| d\theta = 0$$

for all  $p < \infty$ .

*I.e. the domains  $\Omega_r = \Phi\{z: |z| < r\}$  have uniform Lipschitz geometries so that estimates of Dahlberg [8] for the Dirichlet problem et cetera are uniform over  $r$ .*

**Proof.** Follows from the uniform continuity of  $\Phi$ , the argument bounds of Lemma 2.3, the uniform continuity of  $\Psi$ , and the nontangential limits of  $\arg \Phi'$  from Lemma 2.2.  $\square$

2.6. Recall the definition of  $A_p$  weight,  $1 < p < \infty$ . A nonnegative locally integrable function  $w$  satisfies condition  $A_p$  with constant  $c_p$  if for every interval  $I$

$$\frac{1}{|I|} \int_I w \, dx \left( \frac{1}{|I|} \int_I w^{-\frac{1}{p-1}} \, dx \right)^{p-1} \leq c_p.$$

It was proved in [18] that there is then an  $\epsilon > 0$  depending only on  $c_p$  so that  $w \in A_q$  for all  $q > p - \epsilon$ . Applied to  $|\Phi'| \in A_2$  this has the consequence that for some  $p > 1$  small enough,  $|\Phi'|^{-1} \in L^p(\partial\mathbb{D})$  which, since  $\Phi'^{-1} \in H^1(\mathbb{D})$ , implies

$$\Phi'^{-1} \in H^p(\mathbb{D}) \quad (2.5)$$

for  $p > 1$  small enough.

2.7. Returning to the harmonic function  $h$  with gradient nontangentially in  $L^2(\partial\Omega)$ , define  $H(z) = h \circ \Phi(z)$ . Then  $\partial H = \partial h \circ \Phi$ ,  $\Phi'$  is analytic in  $\mathbb{D}$  and, with  $\Omega_r$  as in Theorem 2.4,

$$\begin{aligned} \sup_{r < 1} \int_0^{2\pi} \left| \frac{\partial H(re^{i\theta})}{\Phi'(re^{i\theta})} \right|^2 |\Phi'(re^{i\theta})| \, d\theta &= \lim_{r \rightarrow 1} \int_{\partial\Omega_r} |\partial h|^2 \, ds = \int_{\partial\Omega} |\partial h|^2 \, ds \\ &= \int_{\partial\mathbb{D}} \left| \frac{\partial H}{\Phi'} \right|^2 |\Phi'| \, d\theta < \infty, \end{aligned} \quad (2.6)$$

where Dahlberg's theorem is used in the second equality. (The analogue with 2 replaced by  $p < 2$  is true by estimates for the Neumann problem in plane Lipschitz domains originally due to Fabes and Kenig [15].)

As the counterexamples from the previous section illustrate, the quantity  $\frac{\partial H}{\Phi'}$  directly relates to the continuous vector field  $\vec{\alpha} \circ \Phi$ .

**Lemma 2.5.** For  $\epsilon > 0$  small enough depending only on the Lipschitz geometry of  $\Omega$

$$\frac{\partial H}{\Phi'} \in H^{1+\epsilon}(\mathbb{D}).$$

**Proof.** Hölder's inequality using 2.6 and 2.5.  $\square$

**Remark 2.6.** It is this lemma that fails for the general  $p < 2$  analogue, making possible the above counterexamples. If the exponent  $\frac{1}{p} - 1$  in (1.10) is replaced by  $-1$ , then the quantity  $\frac{\partial H}{\Phi'}$  is obtained (see (1.3)), and its behavior near 1 is then  $(1 - z)^{-1}$  rather than (1.11). Lemma 2.5 does not hold for the elementary examples. On the other hand, if the gradient of  $h$  is nontangentially in  $L^p$  for  $p$  close enough to 2 depending on the Lipschitz constant, then Lemma 2.5 is still true.

2.8. At each point of  $\partial\Omega$  identify the vector field  $\vec{\alpha} = (\alpha_1, \alpha_2)$  with the complex number  $\alpha = \alpha_1 + i\alpha_2$ , and in  $\Omega$  identify  $\nabla h$  with  $2\bar{\partial}h$ . Then the nontangential boundary values are  $\vec{\alpha} \cdot \nabla h = 2\operatorname{Re}(\alpha\bar{\partial}h)$  a.e.(ds). Because  $h$  is not constant the nontangential limits of  $\bar{\partial}h$  cannot vanish on sets of positive measure. Therefore the boundary condition of Theorem 2.1, vis.  $\operatorname{Re}(\alpha\bar{\partial}h) = 0$  a.e.(ds), implies that almost everywhere on  $\partial\Omega$

$$\frac{\bar{\partial}h}{|\bar{\partial}h|} = \pm i\vec{\alpha}$$

or equivalently

$$\frac{\bar{\partial}H}{\Phi'} \frac{|\Phi'|}{|\bar{\partial}H|} = \pm i\vec{\alpha} \circ \Phi \quad \text{a.e.}(d\theta). \quad (2.7)$$

The transverse condition on  $\partial\Omega$ , vis.  $\vec{\alpha} \cdot N \geq c > 0$  a.e.(ds), transferred to  $\partial\mathbb{D}$  reads

$$\operatorname{Re} \frac{\vec{\alpha} \circ \Phi(\theta) e^{i\theta} \Phi'(\theta)}{|\Phi'(\theta)|} \geq c > 0 \quad \text{a.e.}(d\theta). \quad (2.8)$$

**Lemma 2.7.** *There is a measurable set  $E \subset \partial\mathbb{D}$  satisfying  $0 < |E| < 2\pi$  such that*

$$\frac{\bar{\partial}H}{\Phi'} \frac{|\Phi'|}{|\bar{\partial}H|} = \begin{cases} -i\vec{\alpha} \circ \Phi & \text{on } E \text{ a.e.}(d\theta), \\ i\vec{\alpha} \circ \Phi & \text{on } \partial\mathbb{D} \setminus E \text{ a.e.}(d\theta). \end{cases} \quad (2.9)$$

**Proof.** The left-hand side of (2.9) takes only the values (2.7) a.e. If there is no such set  $E$  then, for example, take  $|E| = 0$  in (2.9). By (2.8) the boundary values of  $\operatorname{Re}(-iz\bar{\partial}H(z))$  would be positive a.e.(dθ). But  $\bar{\partial}H \in H^1(\mathbb{D})$  by the Schwarz inequality using (2.6) and (2.4). Consequently  $\operatorname{Im}(z\bar{\partial}H(z))$ , which vanishes at the origin, is the Poisson extension of positive boundary values, a contradiction.  $\square$

2.9. In sum, if  $h$  is not constant, there is an  $H^{1+\epsilon}(\mathbb{D})$  function  $\frac{\bar{\partial}H}{\Phi'}$  satisfying (2.9) for  $\vec{\alpha} \circ \Phi$  continuous on  $\partial\mathbb{D}$ .

2.10. By the canonical factorization of  $H^p$  functions [11, p. 24],  $\frac{\bar{\partial}H}{\Phi'}$  has the *unique* representation in  $\mathbb{D}$

$$\frac{\bar{\partial}H}{\Phi'} = BqG, \quad (2.10)$$

where  $B$  is a Blaschke product,  $q$  is a *singular inner function* and  $G \in H^{1+\epsilon}(\mathbb{D})$  is an *outer function*. In general for any  $p > 0$  an outer function  $\mathcal{O} \in H^p(\mathbb{D})$  has the representation

$$\mathcal{O}(z) = e^{i\theta_0} \exp \left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log |\mathcal{O}(\theta)| d\theta \right\}, \quad (2.11)$$

where  $\theta_0$  is a real number.

By their definitions neither  $q$  nor  $G$  have zeros in  $\mathbb{D}$ . And  $|B(\theta)| = |q(\theta)| = 1$  a.e.(dθ) so that  $|\frac{\bar{\partial}H}{\Phi'}| = |G|$  a.e.(dθ).



**Lemma 2.8.** *Let  $h$  and  $\tilde{\alpha}$  be as in the statement of Theorem 2.1 but with  $h$  not constant. Let  $E$  be the measurable set of Lemma 2.7 satisfying  $0 < |E| < 2\pi$ . Then there is an outer function  $F \in H^{1+\epsilon}(\mathbb{D})$  satisfying (2.9), i.e.*

$$\frac{F}{|F|} = \begin{cases} -i\tilde{\alpha} \circ \Phi & \text{on } E \text{ a.e.}(d\theta), \\ i\tilde{\alpha} \circ \Phi & \text{on } \partial\mathbb{D} \setminus E \text{ a.e.}(d\theta). \end{cases}$$

**Proof.** If  $Bq$  in (2.10) is identically  $-1$  then let  $F = -G$ . If not, then  $Bq + 1$  is an outer function. (The function  $Bq + 1$  is in  $H^\infty$  with positive real part, but more generally any analytic function in  $\mathbb{D}$  with positive real part is outer [11, p. 51].) Consequently  $F = (Bq + 1)^2 G$  is outer. Almost everywhere on  $\partial\mathbb{D}$  the nontangential limits of  $Bq$  and  $(Bq + 1)^2$  have the same principle argument since

$$(Bq + 1)^2 = Bq(Bq + \bar{B}\bar{q} + 2) \quad \text{a.e.}(d\theta).$$

(This device, which was also used in the proof of Theorem 1.1, when applied as here to singular inner functions is apparently Sarason's trick [26].) Thus in either case  $F$  satisfies (2.9) also.  $\square$

2.11. By (2.8) and the continuity of  $\tilde{\alpha}$  the argument of  $\alpha \circ \Phi$  may be taken to be a continuous real valued function  $g^-$  defined on  $\mathbb{R}$  such that  $g^-(\theta + 2\pi) = 2\pi + g^-(\theta)$ . A  $2\pi$ -periodic continuous function may then be defined by  $g(\theta) := \frac{\pi}{2} + \theta - g^-(\theta)$ ,  $-\infty < \theta < \infty$ . Noting that the outer function  $(1 + z)^2$  restricted to  $\partial\mathbb{D}$  is equal to  $e^{i\theta}|1 + e^{i\theta}|^2$ , it follows that

$$i(1 + e^{i\theta})^2 \tilde{\alpha} \circ \Phi(\theta) = |1 + e^{i\theta}|^2 e^{ig(\theta)} \quad \text{everywhere on } \partial\mathbb{D}.$$

Let  $\chi_E$  denote the characteristic function of the set  $E$ . Then by Lemma 2.8 the nontangential limits of the outer function  $(1 + z)^2 F(z) \in H^{1+\epsilon}$  can be expressed a.e.  $(d\theta)$

$$(1 + e^{i\theta})^2 F(\theta) = \exp\{\log|(1 + e^{i\theta})^2 F(\theta)| + i(g(\theta) + \pi\chi_E(\theta))\}, \quad -\pi < \theta \leq \pi, \quad (2.12)$$

with  $g$  continuous on  $\partial\mathbb{D}$ .

## 2.12. End of the proof of Theorem 2.1

By Titchmarsh's theorem [17, p. 58] the Hilbert transform of  $\log|(1 + e^{i\theta})^2 F(\theta)|$  is finite a.e. and the boundary values of the corresponding conjugate harmonic function. Because  $(1 + z)^2 F(z)$  has the representation (2.11) (and by the existence of nontangential limits) it follows from (2.12) that at almost every  $\theta$  this Hilbert transform differs from  $g + \pi\chi_E$  by  $2\pi$  times an integer depending on  $\theta$ . (Replace  $e^{-i\theta_0} F(z)$  by  $F(z)$  and  $g(\theta) - \theta_0$  by  $g(\theta)$ .) Let  $s(\theta)$  denote the simple function on  $\partial\mathbb{D}$  taking only integer values on measurable sets such that the Hilbert transform in question equals  $g + \pi\chi_E + 2\pi s$  a.e. Let  $g$ ,  $\chi_E$  and  $s$  also denote their respective Poisson extensions. The Poisson extension of  $s$  exists because the Poisson extensions of  $g$  and  $\chi_E$  exist. Then the representation (2.11) yields

$$(1 + z)^2 F(z) = \exp\{\log|(1 + z)^2 F(z)| + i(g(z) + \pi\chi_E(z) + 2\pi s(z))\}, \quad z \in \mathbb{D}.$$

The harmonic conjugate of  $2\pi s$  also exists by

$$2\pi \tilde{s}(z) = -\tilde{g}(z) - \pi \tilde{\chi}_E(z) - \log|(1+z)^2 F(z)|.$$

So there is the factorization

$$(1+z)^2 F(z) = \exp\{-\tilde{g}(z) + ig(z)\} \exp\{-\pi \tilde{\chi}_E(z) + i\pi \chi_E(z) - 2\pi \tilde{s}(z) + i2\pi s(z)\}.$$

By using the John–Nirenberg inequality (see, for example, [17, p. 236] or [28, p. 202]) it follows that  $e^{-\tilde{g}} \in L^p(|z| = r)$  uniformly in  $r$  for every  $p < \infty$ . This is because the BMO norm of  $\tilde{g}$  may be assumed as small as one wishes since  $\tilde{g}$  can be approximated by continuous functions in BMO norm. (See, for example, the discussion on p. 221 of Torchinsky's book.) But the same is true for the reciprocal  $e^{\tilde{g}}$ . Consequently by Lemma 2.5 and Hölder's inequality  $\exp\{-\pi \tilde{\chi}_E(z) + i\pi \chi_E(z) - 2\pi \tilde{s}(z) + i2\pi s(z)\}$  is an  $H^1(\mathbb{D})$  function. But an  $H^1(\mathbb{D})$  function that takes only real values a.e. on the boundary is constant. Thus  $(1+z)^{-2} \in H^1(\mathbb{D})$ , a contradiction.

2.13. The contradiction will be made sharper in the next section.

### 3. Uniqueness and the nonhomogeneous problem in the plane

By Calderón's theorem and Theorem 2.1 it follows in the planar case that for the nonhomogeneous oblique derivative problem data need only satisfy one linear condition.

**Definition.** Given a bounded simply connected Lipschitz domain  $\Omega \subset \mathbb{R}^2$  and a continuous transverse unit vector field  $\vec{\alpha}$  on  $\partial\Omega$  let  $f_\alpha$  be the  $L^2(\partial\Omega)$  function of norm one such that the oblique derivative problem can be uniquely solved for any  $L^2$  data orthogonal to  $f_\alpha$ .

In [20, p. 339] it is proved that the  $L^2$  function  $f_\alpha$  may be taken to be *nonnegative* when  $\Omega \subset \mathbb{R}^n$  is Lipschitz and  $\vec{\alpha}$  is the restriction to  $\partial\Omega$  of a  $C^2$  vector field of  $\mathbb{R}^n$ . The proof follows by contradiction from: (1) Nadirashvili's Theorem 7, p. 337, that establishes solvability of the nonhomogeneous problem under these hypotheses for any data  $g \in C(\partial\Omega)$  taken on at each point of the boundary, and (2) Nadirashvili's generalization on p. 327 to Lipschitz domains of the Hopf maximum principle.

To prove the analogous statement here, we need a stronger statement of uniqueness.

**Corollary 3.1.** (See Theorem 2.1.) If  $h$  is harmonic in  $\Omega$  with gradient nontangentially in  $L^2(\partial\Omega)$  and if

$$\text{n.t.} \lim \vec{\alpha} \cdot \nabla h \geq 0 \quad \text{a.e.}(ds),$$

then  $h$  is constant.

**Proof.** Assume  $h$  is not constant. Lemma 2.5 and the arguments preceding it hold under the new hypothesis on  $h$ . Since  $\text{Re}(\alpha \partial h) \geq 0$  a.e. the identity (2.9) is replaced by

$$\frac{\partial H}{\Phi'} \frac{|\Phi'|}{|\partial H|} = e^{i\psi} \vec{\alpha} \circ \Phi \quad \text{a.e.}(d\theta), \quad (3.1)$$

where  $\psi(\theta)$  is a nonconstant real-valued bounded measurable function satisfying  $|\psi| \leq \frac{\pi}{2}$ . The proof of Lemma 2.7 applies to (3.1) to show that  $e^{i\psi}$  cannot be constant. The proof of Lemma 2.8 applies so that  $\frac{\partial H}{\partial \bar{z}}$  in (3.1) can be replaced by an outer function  $F \in H^{1+\epsilon}(\mathbb{D})$ . As in Section 2.11 the outer function  $(1+z)^2 F(z) \in H^{1+\epsilon}$  has nontangential limits a.e.

$$(1 + e^{i\theta})^2 F(\theta) = \exp\{\log|(1 + e^{i\theta})^2 F(\theta)| + i(g(\theta) + \psi(\theta))\}$$

with  $g$  continuous on  $\partial\mathbb{D}$ . The Hilbert transform of  $\log|(1 + e^{i\theta})^2 F(\theta)|$  again differs from  $g + \psi$  by  $2\pi s$  where  $s$  is simple taking integer values on the boundary.

The outer function  $F$  has a square root  $G \in H^{2+2\epsilon}(\mathbb{D})$ . Now departing from the proof of Section 2.12, it follows that

$$\begin{aligned} (1+z)G(z) &= \exp\left\{\log|(1+z)G(z)| + i\left(\frac{1}{2}g(z) + \frac{1}{2}\psi(z) + \pi s(z)\right)\right\} \\ &= \exp\left\{-\frac{1}{2}\tilde{g}(z) + i\frac{1}{2}g(z)\right\} \exp\left\{-\frac{1}{2}\tilde{\psi}(z) + i\frac{1}{2}\psi(z)\right\} \exp\{-\pi\tilde{s}(z) + i\pi s(z)\}. \end{aligned}$$

The reciprocal of the first of the three exponentials is again in  $H^p(\mathbb{D})$  for every  $p < \infty$ . For every  $q < 2$  by the maximum principle (à la Helson–Szegő),  $\exp\{\frac{q}{2}\tilde{\psi}(z)\}$  is dominated by the positive harmonic function  $\sec(\frac{q}{4}\pi) \exp\{\frac{q}{2}\tilde{\psi}(z)\} \cos(\frac{q}{2}\psi(z))$ . Thus the reciprocal of the second exponential is in every  $H^q(\mathbb{D})$ ,  $q < 2$ . Consequently by Hölder's inequality the third exponential is in  $H^1(\mathbb{D})$  and therefore must be constant. Hence  $(1+z)^{-1} \in H^1(\mathbb{D})$  a sharper contradiction.  $\square$

**Theorem 3.2.** Let  $\Omega \subset \mathbb{R}^2$  be a bounded simply connected Lipschitz domain and  $\vec{\alpha}$  a continuous transverse vector field defined on  $\partial\Omega$ . Then for any data  $g \in L^2(\partial\Omega)$  satisfying  $\int_{\partial\Omega} g f_\alpha ds = 0$ , where  $f_\alpha$  is the function defined above, there is a unique, up to constants, solution  $u$  with gradient nontangentially in  $L^2$  so that

$$\text{n.t. } \lim_{X \rightarrow Q} \vec{\alpha}(Q) \cdot \nabla u(X) = g(Q) \quad \text{a.e.}(ds).$$

Furthermore,  $f_\alpha$  can be taken to be nonnegative.

**Proof.** Everything but the last statement follows from Calderón's theorem and Theorem 2.1. If  $f_\alpha$  were to take both positive and negative values on sets of positive measure, it would then follow that the oblique derivative problem could be solved for certain positive data, contradicting the corollary.  $\square$

**Remark 3.3.** As mentioned in Remark 2.6, Lemma 2.5 holds when the gradient of  $h$  is  $L^p$  for  $p$  close enough to 2. This suffices for the proofs of Theorem 2.1 and its corollary. Consequently uniqueness holds for  $p < 2$  depending on the Lipschitz constant for the domain. Calderón's theorem from the introduction also holds for these  $p$ . Therefore  $f_\alpha \in L^{p'}(\partial\Omega)$  for  $p' > 2$  close enough to 2.

#### 4. Infinite-dimensional examples and nonvariational potentials in $\mathbb{R}^n$

A domain and continuous vector field with an infinite-dimensional space of solutions to the homogeneous oblique derivative problem can be constructed by using the well-known examples of solutions to the  $L^p$  Dirichlet problem with vanishing data at reentrant corners. In fact the vector field can be made to be the restriction to the boundary of a  $C^\infty$  vector field defined in the plane.

For example, the harmonic functions

$$u(z) = u_b(z) = \operatorname{Im}\left((iz)^{\frac{1-b}{2-b}}\right), \quad 0 < b < 1,$$

have vanishing  $y$ -derivative on the boundary of the sectors

$$S_b = \left\{ -(3-b)\frac{\pi}{2} < \theta < (1-b)\frac{\pi}{2} \right\}$$

and have gradients nontangentially in  $L^p$  only for  $p < 2 - b$ . Denote by  $S_{b,j}$  similar sectors with vertices at  $v_j = 1 - 2^{-j}$ . Define a bounded sawtooth Lipschitz domain  $\Omega$  to be the intersection of the  $S_{b,j}$  ( $j = 0, 1, 2, \dots$ ) intersected with the disc of radius 3 centered at the origin. The constant vector field  $\alpha = i$  defined in the disc of radius 2, and therefore transverse to the sawtooth part of  $\partial\Omega$ , can be extended to be a  $C^\infty$  unit vector field transverse to all of  $\partial\Omega$ . By Calderón's theorem and any choice of distinct vertices  $v_{j_1}, \dots, v_{j_{N+1}}$  there is some linear combination of the harmonic functions  $u(z - v_{j_k})$  that has oblique data in  $L^2(\partial\Omega)$  satisfying the  $N$  linear conditions sufficient for solvability in the class of solutions with gradients nontangentially in  $L^2$ . The difference, between the  $L^2$  solution for this data and the linear combination that is not an  $L^2$  solution, is a nontrivial solution to the homogeneous problem for every  $p < 2 - b$ . An infinite number of linearly independent solutions are obtained in this way for the same vector field.

Examples for  $p < 2$  of bounded Lipschitz domains and (smooth) continuous vector fields in  $\mathbb{R}^n$  can be constructed in the same way by using known solutions with vanishing Dirichlet data around conical points. Or, for example, any of the above solutions to the homogeneous oblique derivative problem in the plane can be thought of as solutions to the homogeneous problem in an infinite Lipschitz cylinder. The cylinder can then be truncated by conical caps in order to get a bounded domain. The gradients of the cylindrically extended solutions will be nontangentially in  $L^2$  of the conical part of the boundary because of interior regularity (Sobolev imbedding) in the underlying planar domain. The planar vector field can be extended and Calderón's theorem again suffices to produce enough  $L^2$  solutions to get rid of nonzero data. (In the event that the underlying planar domain is not starlike, the tips of the conical part of the domain can be cut off in order to insure that the construction ends with a Lipschitz domain.) Therefore

*In every dimension  $n \geq 2$  and for every  $p < 2$  there exists a bounded Lipschitz domain homeomorphic to the ball that has a (smooth) continuous transverse unit vector field defined on its boundary so that the corresponding  $L^p$  homogeneous oblique derivative problem has an infinite-dimensional space of solutions.*

All of the solutions in these examples are also solutions to the Dirichlet problem with  $W^{1,p}(\partial\Omega)$  data. Therefore by the results in [30] each can be represented by a classical single layer potential

$$Sf(z) = \frac{1}{2\pi} \int_{\partial\Omega} \log|z-w| f(w) ds(w), \quad n=2,$$

or

$$Sf(X) = \frac{1}{(2-n)\omega_n} \int_{\partial\Omega} |X-Q|^{2-n} f(Q) ds(Q), \quad n \geq 3,$$

for a uniquely determined (up to addition of the equilibrium potential)  $L^p(\partial\Omega)$  function  $f$ . As a consequence the nonvariational layer potentials used in [4]

$$\begin{aligned} \text{n.t. } \lim_{X \rightarrow P} \vec{\alpha}(P) \cdot \nabla Sf(X) &= -\frac{1}{2} \vec{\alpha}(P) \cdot N_P f(P) + \text{p.v.} \frac{1}{\omega_n} \int_{\partial\Omega} \frac{\vec{\alpha}(P) \cdot (P-Q)}{|P-Q|^n} f(Q) ds(Q) \\ &= \left( -\frac{1}{2} \vec{\alpha} \cdot N + K_\alpha^* \right) f(P) \end{aligned} \quad (4.1)$$

admit infinite-dimensional kernels in  $L^p(\partial\Omega)$  when  $p < 2$ ,  $\Omega$  is Lipschitz and  $\vec{\alpha}$  is continuous (in fact smooth).

By aid of the Kelvin transform, examples in exterior domains can be constructed by which it then follows that the operators  $\frac{1}{2}\vec{\alpha} \cdot N + K_\alpha^*$  admit infinite-dimensional kernels. The operators  $\frac{1}{2}\vec{\alpha} \cdot N + K_\alpha$  dual to these arise from computing the Dirichlet boundary values from the interior of  $\Omega$  of the potentials

$$\frac{1}{\omega_n} \int_{\partial\Omega} \frac{\vec{\alpha}(Q) \cdot (Q-X)}{|Q-X|^n} f(Q) ds(Q).$$

They are the nonvariational analogues of the classical double layer potentials, but unlike these latter operators [9] have in general infinite-dimensional cokernel for  $p > 2$ . As observed in [16] neither can they generally be Fredholm operators for  $p < 2$ . If they were, a priori  $L^p$  estimates  $p < 2$  for the Dirichlet problem would hold for  $L^2(\partial\Omega)$  solutions to the Dirichlet problem leading to a contradiction by duality like that in [10, pp. 131, 132]. Thus the nonvariational layer potentials that arise from continuous transverse vector fields on Lipschitz boundaries cannot be Fredholm outside an (a small) interval of  $p$ 's about 2.

On the other hand, Theorem 2.1 and the results in [33] for Hölder vector fields in higher dimensions show that Calderón's nonvariational potentials can be as well behaved as the classical when  $p = 2$ . Generalizations may be a useful substitute for potentials based on the normal vector field when formulating integral equations for higher order boundary value problems.

## 5. $L^2$ examples and $p \leq 2$ uniqueness for perturbations of the normal vector field

By letting the parameter in the definition of  $F_b$  (1.4) vary from that of the conformal map  $\Phi_b$  (1.9) one can obtain nontrivial solutions to the  $L^2$  homogeneous oblique problem in curvilinear polygons with respect to discontinuous vector fields.

Let  $0 < \tilde{b} < 1$  denote the parameter in (1.4). Then, for example at the point 1 on the boundary of the disc,  $F_{\tilde{b}}$  behaves as  $(1-z)^{-\tilde{b}}$  and  $\Phi'_{\tilde{b}}$  as  $(1-z)^{1-b}$  so that  $F_{\tilde{b}}(z)(\Phi'_{\tilde{b}}(z))^{\frac{1}{2}-1}$  will be in  $H^2(\mathbb{D})$  when  $-\tilde{b} - \frac{1}{2}(1-b) > -\frac{1}{2}$ . Thus

$$0 < \tilde{b} < \frac{b}{2}$$

and  $\Omega = \Phi_b(\mathbb{D})$  now imply that  $G' = \frac{F_{\tilde{b}}}{z\Phi'_b} \circ \Phi_b^{-1}$  is nontangentially in  $L^2(\partial\Omega)$ . Allowing  $0 < b < 2$ , the domain  $\Omega$  is as described in the elementary example when  $0 < b \leq 1$  and it is the intersection of the two discs described in the elementary example when  $1 \leq b < 2$ . In either case the opening of each of the corners of  $\Omega$  is  $\pi(2-b)$ . Define the vector field  $\vec{\alpha}$  by (1.7) (or, which is the same thing, by (1.13) using  $G'$ ). The vector field is now discontinuous at the corners with jump in argument equal to  $(b-\tilde{b})\pi$ . The transverse condition will still be described by (1.5) so that

$$\inf_{\partial\Omega} N \cdot \vec{\alpha} = \sin \tilde{b} \frac{\pi}{2}. \quad (5.1)$$

(In the example corresponding to  $b = 1$ , i.e. no corners,  $N \cdot \vec{\alpha}$  takes values less than  $\frac{1}{\sqrt{2}}$  corroborating Nadirashvili's uniqueness condition (5') [21] for the  $L^2$  problem in smooth domains.)  $G'$  is the analytic representation of  $\nabla h$  which is orthogonal to  $\vec{\alpha}$  almost everywhere on  $\partial\Omega$ .

The examples just described suggest that if the transverse condition for  $\vec{\alpha}$  is

$$\inf N \cdot \vec{\alpha} > \sin b \frac{\pi}{4} = \cos \left( 1 - \frac{b}{2} \right) \frac{\pi}{2}$$

when  $1 \leq b < 2$  and the smallest opening at any corner point is  $\pi(2-b)$ , then the  $L^2$  homogeneous oblique derivative problem should have only the constant solution. Or put in terms of a Lipschitz constant  $M$ , a condition implying uniqueness should be

$$\inf N \cdot \vec{\alpha} > \cos \frac{1}{2} \left( \frac{\pi}{2} - \arctan M \right). \quad (5.2)$$

This condition for uniqueness is in fact the case and violating it allows the above examples of nonuniqueness.

**Theorem 5.1.** *Let  $\Omega \subset \mathbb{R}^2$  be a bounded Lipschitz domain conformally equivalent to the disc by  $\Phi: \mathbb{D} \rightarrow \Omega$ . Let  $\Psi$  and  $\delta > 0$  be as in Lemma 2.3 and let  $0 < (2-b)\frac{\pi}{2} < \frac{\delta}{2}$ . Let  $\vec{\alpha}$  be a measurable unit vector field defined on  $\partial\Omega$  satisfying the transverse condition*

$$\vec{\alpha} \cdot N \geq \cos \left( 1 - \frac{b}{2} \right) \frac{\pi}{2}$$

almost everywhere. Let  $h$  be harmonic in  $\Omega$  and nontangentially in  $L^2(\partial\Omega)$ . Then

$$\text{n.t.} \lim \vec{\alpha} \cdot \nabla h \geq 0 \quad \text{a.e.}(ds) \quad (5.3)$$

implies that  $h$  is constant.

**Proof.** The proof is a modification of the proofs of Theorem 2.1 and Corollary 3.1. By the transverse condition there is a measurable function  $|\lambda(\theta)| \leq (1 - \frac{b}{2})\frac{\pi}{2}$  defined on  $\partial\mathbb{D}$  so that

$$\alpha \circ \Phi(\theta) = e^{i\lambda(\theta)} e^{i\theta} \frac{\Phi'(\theta)}{|\Phi'(\theta)|}.$$

By (1.2) and (5.3)

$$\frac{\partial H}{\partial \Phi'} \frac{|\Phi'|}{|\partial H|} = e^{i\psi(\theta)} e^{-i\lambda(\theta)} e^{-i\theta} \frac{\overline{\Phi'(\theta)}}{|\Phi'(\theta)|} \quad \text{a.e.}(d\theta),$$

where  $|\psi| \leq \frac{\pi}{2}$  as in (3.1). This simplifies to

$$\frac{\partial H}{|\partial H|} = e^{i\psi(\theta)} e^{-i\lambda(\theta)} e^{-i\theta} \quad \text{a.e.}(d\theta). \quad (5.4)$$

The Helson–Szegő argument used in the proof of Corollary 3.1 implies that the analytic function  $\Lambda = e^{-\tilde{\lambda} + i\lambda}$  is in  $H^q(\mathbb{D})$  for every  $q < \frac{2}{2-b}$ . Likewise  $\frac{\Phi'}{\Psi} \in H^{\frac{1}{b-1} + \epsilon}(\mathbb{D})$  for  $\epsilon > 0$  small enough and therefore  $\Phi' \in H^{\frac{1}{b-1} + \epsilon}$  also. Consequently  $\partial H \in H^r(\mathbb{D})$  for some  $r > \frac{2}{b}$  by Hölder's inequality and (2.6). As in Sections 2.10 and 2.11  $\partial H$  can be replaced with an outer function  $F \in H^r(\mathbb{D})$  and  $(1+z)^2$  introduced. By the relations between  $q$ ,  $r$ , and the dual exponents  $\frac{2}{2-b}$  and  $\frac{2}{b}$  the outer function  $(1+z)^2 F(z) \Lambda(z)$  is in  $H^{1+\epsilon}(\mathbb{D})$  and has boundary values

$$|1 + e^{i\theta}|^2 |F(\theta)| |\Lambda(\theta)| e^{i\psi(\theta)} \quad \text{a.e.}(d\theta)$$

by (5.4). Now the square root may be employed and the argument finished as in the proof of Corollary 3.1.  $\square$

The same argument applies to other  $p$ .

**Corollary 5.2.** Let  $p > 1$  and  $0 < (2-b)\frac{\pi}{2} < \frac{\delta}{2}$ . Suppose  $\nabla h$  is nontangentially in  $L^p$  and satisfies (5.3). If

$$\vec{\alpha} \cdot N \geq \cos \frac{p-1}{p} (2-b) \frac{\pi}{2}$$

then  $h$  is constant.

In the two parameter setup of the elementary example a corner opening of  $(2-b)\pi$  allows a nontrivial solution to the homogeneous oblique problem for  $L^p$  when  $\tilde{b} < \frac{1}{p} + (b-1)\frac{p-1}{p}$ , i.e. from (5.1)  $\sin \tilde{b}\frac{\pi}{2} = \cos(1-\tilde{b})\frac{\pi}{2} < \cos \frac{p-1}{p} (2-b)\frac{\pi}{2}$ .

## 6. Questions

- (1) In higher dimensions Hölder continuity of the vector field will be shown [33] to imply that the homogeneous problem has only the constant solution. What if the vector field is merely continuous?
- (2) The vector fields of Sections 1 and 4 that lead to nonuniqueness are rather nice. Therefore, given a Lipschitz domain and a  $p < 2$  is it possible to construct a continuous vector field for which the  $L^p$  homogeneous problem has only the constant solution? If the answer is yes, the corresponding layer potentials still will not have closed range. For example in the plane, if  $\frac{1}{2}\vec{\alpha} \cdot N + K_\alpha^*$  is one-to-one with closed range when  $p < 2$ , it follows that  $\frac{1}{2}\vec{\alpha} \cdot N + K_\alpha$  is invertible for  $p' > 2$  since  $\frac{1}{2}\vec{\alpha} \cdot N + K_\alpha$  is one-to-one in  $L^2(\partial\Omega)$  (by a variation of Theorem 2.1). As Calderón argues in [4]  $K_\alpha + K_\alpha^*$  is compact when  $\vec{\alpha}$  is continuous. Consequently  $-\frac{1}{2}\vec{\alpha} \cdot N + K_\alpha^*$  would have index zero for  $p' > 2$  as would  $-\frac{1}{2}\vec{\alpha} \cdot N + K_\alpha$  for  $p < 2$  leading to a contradiction like that mentioned in Section 4.
- (3) Can the nonhomogeneous problem be solved for data in  $L^p(\partial\Omega)$  when the vector field is a perturbation of the normal under the constraints of Corollary 5.2? The constraints show that only the classical Neumann problem can be considered for solution when data is prescribed in the Hardy space  $H^1(\partial\Omega)$ . For  $p = 2$  and  $n \geq 2$  the condition (5.2) is necessary and sufficient for the model Rellich identity

$$\int_{\partial\Omega} |\nabla u|^2 N \cdot \vec{e} \, ds = 2 \int_{\partial\Omega} \frac{\partial u}{\partial N} \vec{e} \cdot u \, ds$$

to yield control of the gradient in terms of  $\vec{\alpha} \cdot \nabla u$ . (Here  $\vec{e}$  is perpendicular to the hyperplane over which  $\partial\Omega$  is a Lipschitz graph.) This is seen by decomposing  $N$  and  $\vec{e}$  on the right into components in the  $\vec{\alpha}$  direction and perpendicularly. Only the resulting term (of four) without an  $\vec{\alpha}$  derivative will require hiding on the left. This will be possible if  $2 \sin(\alpha, N) \sin(\alpha, \vec{e}) < N \cdot \vec{e}$  which is implied by (5.2).

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